and $\gamma = 1.4$ were essentially independent of Mach number and are plotted as a single curve in Fig. 7b.

Reference 4 gives numerical results for the density at the center of an initially uniform spherical gas cloud expanding to a vacuum for $\gamma = \frac{5}{3}$. The results are presented by a plot of ρ vs t which yields $D = 0.38$. The latter value can be compared with the approximate value $D = 0.434$ given in Table 1 for $\sigma = 2$, $\gamma = \frac{5}{3}$. However, it should be noted that the initial portion of the continuum curve of ρ vs t in Ref. 4 obviously is incorrect, since the numerical solution for ρ vs t approaches the asymptotic solution from below rather than from above (compare with Fig. 7). This casts some doubt on the correctness of the resulting value $D = 0.38$.

In general, it appears that the value of D given by the approximate solution [Eq. (28)] tends to be somewhat too large for $\sigma = 1, 2$, although it is exact for $\sigma = 0$ and N an integer. The two numerical solutions noted herein indicated that D is too large by about 10%. Further numerical solutions are required to define better the accuracy of the approximate solution.#

References

[1] Sedov, L. I., *Similarity and Dimensional Methods in Mechanics* (Academic Press, New York, 1959), pp. 271–281.

[2] Keller, J. B., "Spherical, cylindrical, and one dimensional gas flows," Quart. Appl. Math. **14**, 171–184 (1956).

[3] Courant, R. and Friedricks, K. O., *Supersonic Flow and Shock Waves* (Interscience Publishers, New York, 1948), pp. 191–195.

[4] Molmud, P., "Expansion of a rarified gas cloud into a vacuum," Phys. Fluids **3**, 362–366 (1960).

[5] Greifinger, C. and Cole, J., "One dimensional expansion of a finite mass of gas into vacuum," Rand Rept. P2008 (June 6, 1960).

[6] Stanyukovich, K. P., *Unsteady Motion of Continuous Media* (Pergamon Press, New York, 1960), pp. 147–163, 498–506.

[7] Greenspan, H. P. and Butler, D. S., "On the expansion of a gas into vacuum," J. Fluid Mech. **13**, 101–119 (1962).

After completion of the present report, the authors communicated with C. Greifinger, of Rand Corporation, who has obtained numerical solutions for the continuum expansion of an initially uniform gas into a vacuum. These results are as yet unpublished. However, Dr. Greifinger has compared the values of D, in Table 1, with his results. He stated that, for $\sigma = 1$, the values of D for $\gamma = \frac{11}{9}$ and $\gamma = 1.4$ are about 10% and 20% high, respectively. For $\sigma = 2$, he has stated that the values of D for $\gamma = \frac{11}{9}$ and $\gamma = \frac{5}{3}$ are about 50% and 100% high, respectively. Thus the approximate solution of the present report appears to become more in error as σ and γ increase. However, for hypersonic rocket exhausts ($\sigma = 1$, $\gamma \approx 1.2$), the approximate solution should be correct to within 10%. The publication of Dr. Greifinger's results will permit a more accurate evaluation of the approximate solution and might provide a basis for improvement.

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Powered Flight Trajectories of Rockets under Oriented Constant Thrust

Chong-Hung Zee*
Curtiss-Wright Corporation, Wood-Ridge, N. J.

The second-order nonlinear differential equations of motion in the case of a rocket in drag-free powered flight under a thrust of constant magnitude and fixed orientation are solved by series expansions developed to the seventh power of the independent variable "time." The coefficients of each power of "time" are in terms of the preceding ones and, consequently, in terms of the initial conditions. The truncation errors of the series are estimated; hence their accuracy can be evaluated. The case of oriented constant thrust acceleration also is included in the present analysis.

Nomenclature

A	$= \dot{m}\bar{c}/M_0g_0$
$a_0, a_1, a_2, \ldots$	= constants to be determined
B	$= (\dot{m}/M_0)(r_0/g_0)^{1/2}$
$b_0, b_1, b_2, \ldots$	= constant to be determined
C	$= (d\rho/d\tau)_0$
C'	$= (dr/dt)_0$
$\bar{c}$	= effective average exhaust velocity of the jet
D	$= (d\theta/d\tau)_0$
D'	$= (d\theta/dt)_0$
g_0	= gravitational constant at distance r_0 from the center of attraction
M	$= \cos\psi$
M_0	= mass of rocket at the beginning of thrusting
$\dot{m}$	= constant flow rate of propellant mass
N	$= \sin\psi$
r	= distance between the rocket and the center of attraction at any time t
r_0	= distance between the rocket and the center of attraction at $t = 0$
t	= time
V_x	= velocity component in the X direction
V_y	= velocity component in the Y direction
X	= coordinate (origin at $r = r_0$, $\theta = 0$)
Y	= coordinate (origin at $r = r_0$, $\theta = 0$)
$Rn1, Rn2, \ldots$	= remainder of truncated series
θ	= angle between radius vectors r_0 and r
$\xi_1, \xi_2, \ldots, \xi_{1m}, \xi_{2m}$	= values of τ between the interval 0 and τ
ρ	$= r/r_0$
τ	$= (g_0/r_0)^{1/2}t$
ψ	= angle between the thrust vector and the radius vector r_0

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* Project Engineer, Engineering Department, Wright Aeronautical Division.

Subscripts

0 = quantity at $t = 0$
T = truncated series value

Introduction

In dealing with powered flight trajectories, most of the existing literatures are limited to the case of constant thrust acceleration with a specified program of thrusting direction such as radial, circumferential (normal to the radius), tangential, etc.[1-3] For orbital transfers of the Hohmann type, near impulsive thrusting has been investigated,[4] which is again limited to the case of constant thrust acceleration; as a matter of practical importance, the constant thrust powered flight trajectories for Hohmann-type transfers have been investigated recently by Zee.[5] In all the forementioned investigations, a rocket is assumed to be a particle.

For a liquid propellant rocket, the thrust is usually constant and the thrust-on time is short. If a certain specified program of thrusting direction is required, the rocket should have either attitude controls or gimbals accordingly during the thrusting period. In practice, the attitude controls and gimbals may be difficult to carry out because the time is so short. Thus, in maneuvering a rocket in the space, the most general, simple, and practical case is to thrust the rocket in a desired direction with a constant thrusting force. In this case, during the powered flight the rocket is kept in a fixed orientation, and the angle between the radius vectors at the beginning and at the end of the powered flight is small because the thrust-on time is short. Hence, in any coplanar orbital transfer, a rocket is required to orient itself in the desired direction by means of attitude control before it reaches the transfer point.

In studying the trajectory optimization for ballistic missiles, it has been found that a constant or nearly constant thrust attitude maximizes the range.[6] During the powered flight, the external force field is assumed to be constant both in magnitude and direction (flat earth approximation), which in turn limits the powered arc range angle to be small. However, the burnout conditions (both the velocity and the location) are related closely to the free flight followed and hence to the range; thus a comparison of burnout conditions obtained from flat earth approximation and from the present analysis should provide valuable information in finding the range of a ballistic missile. It is hoped that the analysis presented in this paper will be employed in the study of the effect of finite thrusting time in orbital maneuvers, which, in the case of Hohmann-type transfer, has been well explored.[5]

Analysis

The equations of motion in polar coordinates for a rocket in a drag-free flight under oriented constant thrust and a single center of attraction are (see Fig. 1)†

$$\frac{d^2r}{dt^2} = \frac{\dot{m}\bar{c}}{M_0 - \dot{m}t}\cos(\psi - \theta) + r\left(\frac{d\theta}{dr}\right)^2 - \frac{g_0r_0^2}{r^2} \tag{1}$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\dot{m}\bar{c}}{M_0 - \dot{m}t}r\sin(\psi - \theta) \tag{2}$$

Letting $\rho = r/r_0$, $\tau = (g_0/r_0)^{1/2}t$, $A = \dot{m}\bar{c}/M_0g_0$, and $B = (\dot{m}/M_0)(r_0/g_0)^{1/2}$, Eqs. (1) and (2) become

$$\frac{d^2\rho}{d\tau^2} = \frac{A}{1 - B\tau}\cos(\psi - \theta) + \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} \tag{3}$$

$$\frac{d}{d\tau}\left(\rho^2\frac{d\theta}{d\tau}\right) = \frac{A}{1 - B\tau}\rho\sin(\psi - \theta) \tag{4}$$

† $\dot{m}\bar{c}$ is always positive, and the sign of the entire term is governed automatically by the angle ψ.

Expand

$$\cos(\psi - \theta) = \cos\psi\cos\theta + \sin\psi\sin\theta$$
$$\sin(\psi - \theta) = \sin\psi\cos\theta - \cos\psi\sin\theta$$

and approximate

$$\sin\theta = \theta - \tfrac{1}{6}\theta^3$$
$$\cos\theta = 1 - \tfrac{1}{2}\theta^2$$

because the total subtended angle of the powered flight trajectory is small, as stated previously. Hence, Eqs. (3) and (4) become

$$\frac{d^2\rho}{d\tau^2} = \frac{A}{1 - B\tau}\left[M\left(1 - \frac{1}{2}\theta^2\right) + N\left(\theta - \frac{1}{6}\theta^3\right)\right] + \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} \tag{5}$$

$$\frac{d}{d\tau}\left(\rho^2\frac{d\theta}{d\tau}\right) = \frac{A}{1 - B\tau}\rho\left[N\left(1 - \frac{1}{2}\theta^2\right) - M\left(\theta - \frac{1}{6}\theta^3\right)\right] \tag{6}$$

where $N = \sin\psi$ and $M = \cos\psi$ are known, for ψ is one of the given initial conditions.

The initial conditions for Eqs. (1) and (2) are $(dr/dt)_0 = C'$, $(d\theta/dt)_0 = D'$, and $\theta = 0$ at $t = 0$, $r = r_0$; and the corresponding initial conditions for Eqs. (5) and (6) are $(d\rho/d\tau)_0 = C$, $(d\theta/d\tau)_0 = D$, and $\theta = 0$ at $\tau = 0$, $\rho = 1$.

Set

$$\rho = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5 + a_6\tau^6 + a_7\tau^7 + \ldots \tag{7}$$

$$\theta = b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + b_5\tau^5 + b_6\tau^6 + b_7\tau^7 + \ldots \tag{8}$$

then

$$d^2\rho/d\tau^2 = 2a_2 + 6a_3\tau + 12a_4\tau^2 + 20a_5\tau^3 + 30a_6\tau^4 + 42a_7\tau^5 + \ldots \tag{9}$$

$$d\theta/d\tau = b_1 + 2b_2\tau + 3b_3\tau^2 + 4b_4\tau^3 + 5b_5\tau^4 + 6b_6\tau^5 + 7b_7\tau^6 + \ldots \tag{10}$$

and expand

$$A/(1 - B\tau) = A(1 + \overline{B\tau} + \overline{B\tau^2} + \overline{B\tau^3} + \overline{B\tau^4} + \ldots) \tag{11}$$

Note that the coefficients a_0, a_1, b_0, and b_1 can be determined directly from the initial conditions:

$$\rho = 1 \text{ at } \tau = 0,\ a_0 = 1$$
$$(d\rho/d\tau)_0 = C \text{ at } \tau = 0,\ a_1 = C$$
$$\theta = 0 \text{ at } \tau = 0,\ b_0 = 0$$
$$(d\theta/d\tau)_0 = D \text{ at } \tau = 0,\ b_1 = D$$

Substituting the series of Eqs. (7–11) into Eq. (5) by means of the usual techniques in power series operations, the following relationships are obtained:

$$\begin{aligned}
2a_2 &= AM + b_1^2 - 1 \\
6a_3 &= A(BM + Nb_1) + a_1(b_1^2 + 2) + 4b_1b_2 \\
12a_4 &= A(B^2M + BNb_1 + Nb_2 - \tfrac{1}{2}Mb_1^2) + a_2(b_1^2 + 2) + 4a_1b_1b_2 + 4b_2^2 + 6b_1b_3 - 3a_1^2 \\
20a_5 &= A(B^3M + B^2Nb_1 + BNb_2 - \tfrac{1}{2}MBb_1^2 + Nb_3 - Mb_1b_2 - \tfrac{1}{6}Nb_1^3) + a_3(b_1^2 + 2) + 4a_2b_1b_2 + a_1(4b_2^2 + 6b_1b_3) + (8b_1b_4 + 12b_2b_3) - 6a_1a_2 + 4a_1^3 \\
30a_6 &= A[B^4M + B^3Nb_1 + B^2Nb_2 - \tfrac{1}{2}MB^2b_1^2 + B(Nb_3 - Mb_1b_2 - \tfrac{1}{6}Nb_1^3) + Nb_4 - \tfrac{1}{2}M(b_2^2 + 2b_1b_3) - \tfrac{1}{2}Nb_1^2b_2] + a_4(b_1^2 + 2) + 4a_3b_1b_2 + a_2(4b_2^2 + 6b_1b_3) + a_1(8b_1b_4 + 12b_2b_3) + (9b_3^2 + 16b_2b_4 + 10b_1b_5) + 12a_1^2a_2 - 5a_1^4 - 6a_1a_3 - 3a_2^2
\end{aligned} \tag{12}$$

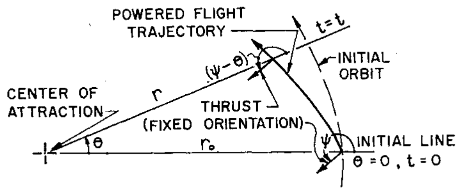

Fig. 1 Notations

$$
\begin{aligned}
42a_7 = {} & A[B^5M + B^4Nb_1 + B^3(Nb_2 - \tfrac{1}{2}Mb_1^2) + \\
& B^2(Nb_3 - Mb_1b_3 - \tfrac{1}{6}Nb_1^3) + \\
& B(Nb_4 - \tfrac{1}{2}Mb_2^2 - Mb_1b_3 - \tfrac{1}{2}Nb_1^2b_2) + \\
& Nb_5 - M(b_1b_4 + b_2b_3) - \tfrac{1}{2}N(b_1^2b_3 + b_1b_2^2)] + \\
& a_5(b_1^2 + 2) + 4a_4b_1b_2 + a_3(4b_2^2 + 6b_1b_3) + \\
& a_2(8b_1b_4 + 12b_2b_3) + a_1(9b_3^2 + 16b_2b_4 + \\
& 10b_1b_5) + (12b_1b_6 + 20b_2b_5 + 24b_3b_4) - \\
& 20a_1^3a_2 + 6a_1^5 + 12a_1^2a_3 + 12a_1a_2^2 - \\
& 6a_1a_4 - 6a_2a_3
\end{aligned}
$$

Again the same technique is employed by substituting the series of Eqs. (7, 8, and 11) into Eq. (6), and integrating the right side of Eq. (6) term by term with the initial condition $(d\theta/d\tau)_0 = D$ at $\rho = 1$, $\tau = 0$ yields

$$
\begin{aligned}
& 2b_2 + 2a_1b_1 = AN \\
& 3b_3 + 4b_1b_2 + b_1(a_1^2 + 2a_2) = \tfrac{1}{2}A[N(B + a_1) - Mb_1] \\
& 4b_4 + 6a_1b_3 + 2b_2(a_1^2 + 2a_2) + b_1(2a_3 + 2a_1a_2) = \\
& \quad \tfrac{1}{3}A[N(B^2 + a_1B + a_2) - Mb_2 - \tfrac{1}{2}Nb_1^2 - \\
& \qquad Mb_1(B + a_1)] \\
& 5b_5 + 8a_1b_4 + 3b_3(a_1^2 + 2a_2) + 4b_2(a_3 + a_1a_2) + \\
& \quad b_1(a_2^2 + 2a_4 + 2a_1a_3) = \tfrac{1}{4}A[N(B^3 + a_1B^2 + \\
& \quad a_2B + a_3) - Mb_1(B^2 + a_1B + a_2) + \\
& \quad \tfrac{1}{6}Mb_1^3 - (Mb_2 + \tfrac{1}{2}Nb_1^2)(B + a_1) - Mb_3 - Nb_1b_2] \\
& 6b_6 + 10a_1b_5 + 4b_4(a_1^2 + 2a_2) + 6b_3(a_3 + a_1a_2) + \\
& \quad 2b_2(a_2^2 + 2a_4 + 2a_1a_3) + 2b_1(a_5 + a_1a_4 + a_2a_3) = \\
& \quad \tfrac{1}{5}A[N(B^4 + a_1B^3 + a_2B^2 + a_3B + a_4) - \qquad (13) \\
& \quad Mb_1(B^3 + a_1B^2 + a_2B + a_3) - \\
& \quad (Mb_2 + \tfrac{1}{2}Nb_1^2)(B^2 + a_1B + a_2) + \\
& \quad (\tfrac{1}{6}Mb_1^3 - Mb_3 - Nb_1b_2)(B + a_1) - \\
& \quad Mb_4 - \tfrac{1}{2}N(b_2^2 + 2b_1b_3) + \tfrac{1}{2}Mb_1^2b_2] \\
& 7b_7 + 12a_1b_6 + 5b_5(a_1^2 + 2a_2) + 8b_4(a_3 + a_1a_2) + \\
& \quad 3b_3(a_2^2 + 2a_4 + 2a_1a_3) + 4b_2(a_5 + a_1a_4 + a_2a_3) + \\
& \quad b_1(a_3^2 + 2a_1a_5 + 2a_2a_4 + 2a_6) = \\
& \quad \tfrac{1}{6}A\{N(B^5 + a_1B^4 + a_2B^3 + a_3B^2 + a_4B + a_5) - \\
& \quad Mb_1(B^4 + a_1B^3 + a_2B^2 + a_3B + a_4) - \\
& \quad (Mb_2 + \tfrac{1}{2}Nb_1^2)(B^3 + a_1B^2 + a_2B + a_3) + \\
& \quad (\tfrac{1}{6}Mb_1^3 - Mb_3 - Nb_1b_3)(B^2 + a_1B + a_2) + \\
& \quad [\tfrac{1}{2}Mb_1^2b_2 - Mb_4 - \tfrac{1}{2}N(b_2^2 + 2b_1b_3)](B + a_1) - \\
& \quad Mb_5 - N(b_1b_4 + b_2b_3) + \tfrac{1}{2}M(b_1^2b_3 + b_1b_2^2)\}
\end{aligned}
$$

As is seen in Eqs. (12) and (13), all a's and b's are in terms of the preceding ones and, consequently, can be determined one after another. Eqs. (7) and (8) describe completely the powered flight trajectory of a rocket under oriented constant thrust; their first derivatives give the corresponding radial and angular velocities of the rocket.

Table 1 Various elements of powered flight trajectory at $\tau = 0.1$

Nondimensional value		Actual value	
ρ_T	0.9945 43238	r_T	22,596,022.4 ft.
ρ_T'	−0.1142 08466	$\left.\frac{dr}{dt}\right\|_T$	−2841.5 fps
ρ_T''	−1.3063 80654	$\left.\frac{d^2r}{dt^2}\right\|_T$	−35.6 ft/sec²
θ_T	0.1005 49679	θ_T	0.1005 49679 rad
θ_T'	1.0170 84518	$\left.\frac{d\theta}{dt}\right\|_T$	1.1138×10^{-3} rad/sec
θ_T''	0.3677 11010	$\left.\frac{d^2\theta}{dt^2}\right\|_T$	0.4410×10^{-6} rad/sec²

Table 2 Various errors of powered flight trajectory at $\tau = 0.1$

	Truncation error only [from Eqs. (22) and (24)]		Truncation and approximation errors [from Eqs. (26) and (27)]	
Rn's	Non-dimensional value	Actual value	Non-dimensional value	Actual value
$Rn1$	-49×10^{-9}	−1.1 ft	-51×10^{-9}	−1.2 ft
$Rn2$	−3920	−0.1 fps	−4065	−0.1 fps
$Rn3$	−274420	−0.01 ft/sec²	−28456	−0.01 ft/sec²
$Rn4$	45	45×10^{-9} rad	45	45×10^{-9} rad
$Rn5$	3602	3.9×10^{-9} rad/sec	3605	3.9×10^{-9} rad/sec
$Rn6$	252115	0.3×10^{-9} rad/sec²	252395	0.3×10^{-9} rad/sec²

Accuracy

Because of the complicated expression of the a's and b's, it is impractical to develop further the series in Eqs. (7) and (8); then the series in their truncated forms yield the results with truncation errors. An estimate of these errors is of importance to the results obtained from the present analysis. Consider the following series:

$$\rho = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5 + a_6\tau^6 + a_7\tau^7 + Rn1 = \rho_T + Rn1 \qquad (14)$$

$$d\rho/d\tau = a_1 + 2a_2\tau + 3a_3\tau^2 + 4a_4\tau^3 + 5a_5\tau^4 + 6a_6\tau^5 + 7a_7\tau^6 + Rn2 = \rho_T' + Rn2 \qquad (15)$$

$$d^2\rho/d\tau^2 = 2a_2 + 6a_3\tau + 12a_4\tau^2 + 20a_5\tau^3 + 30a_6\tau^4 + 42a_7\tau^5 + Rn3 = \rho_T'' + Rn3 \qquad (16)$$

$$\theta = b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + b_5\tau^5 + b_6\tau^6 + b_7\tau^7 + Rn4 = \theta_T + Rn4 \qquad (17)$$

$$d\theta/d\tau = b_1 + 2b_2\tau + 3b_3\tau^2 + 4b_4\tau^3 + 5b_5\tau^4 + 6b_6\tau^5 + 7b_7\tau^6 + Rn5 = \theta_T' + Rn5 \qquad (18)$$

$$d^2\theta/d\tau^2 = 2b_2 + 6b_3\tau + 12b_4\tau^2 + 20b_5\tau^3 + 30b_6\tau^4 + 42b_7\tau^5 + Rn6 = \theta_T'' + Rn6 \qquad (19)$$

If these series are conceived as Taylor's series, then the remainders, Rn's, in the Lagrangian form are‡

$$
\begin{aligned}
Rn1 &= (1/8!)\rho^{\mathrm{VIII}}(\xi_1)\tau^8 & 0 < \xi_1 < \tau \\
Rn2 &= (1/7!)\rho^{\mathrm{VIII}}(\xi_2)\tau^7 & 0 < \xi_2 < \tau \\
Rn3 &= (1/6!)\rho^{\mathrm{VIII}}(\xi_3)\tau^6 & 0 < \xi_3 < \tau \\
Rn4 &= (1/8!)\theta^{\mathrm{VIII}}(\xi_4)\tau^8 & 0 < \xi_4 < \tau \\
Rn5 &= (1/7!)\theta^{\mathrm{VIII}}(\xi_5)\tau^7 & 0 < \xi_5 < \tau \\
Rn6 &= (1/6!)\theta^{\mathrm{VIII}}(\xi_6)\tau^6 & 0 < \xi_6 < \tau
\end{aligned}
$$

Although all ξ's depend to some extent on the magnitude of τ if τ is small, the following approximations should be true:

$$
\begin{aligned}
\rho^{\mathrm{VIII}}(\xi_1) = \rho^{\mathrm{VIII}}(\xi_2) = \rho^{\mathrm{VIII}}(\xi_3) = \rho^{\mathrm{VIII}}(\xi_{1m}) \qquad 0 < \xi_{1m} < \tau \\
\theta^{\mathrm{VIII}}(\xi_4) = \theta^{\mathrm{VIII}}(\xi_5) = \theta^{\mathrm{VIII}}(\xi_6) = \theta^{\mathrm{VIII}}(\xi_{2m}) \qquad 0 < \xi_{2m} < \tau
\end{aligned}
$$

where $\rho^{\mathrm{VIII}}(\xi_{1m})$ and $\theta^{\mathrm{VIII}}(\xi_{2m})$ are the mean values of $\rho^{\mathrm{VIII}}(\tau)$

‡ Roman numerals denote order of derivatives.

Table 3 Comparison of burnout conditions and total range angle

Case	Burnout conditions: Velocity, fps V_x	V_y	Location, ft X	Y	Total range angle, deg
I. Present analysis	−24754.9	−5353.5	−2,268,196.3	−238,107.0	56.6
II. Flat earth approximation	−24880.3	−5348.9	−2,272,000.0	−238,003.7	62.7
III. Impulsive thrust	−24880.3	−2860.9	0	0	49.2

and $\theta^{\text{VIII}}(\tau)$, respectively, in the interval $(0,\tau)$. Hence,

$$Rn1 = \tfrac{1}{8}\tau Rn2 = \tfrac{1}{56}\tau^2 Rn3 \quad (20)$$

$$Rn4 = \tfrac{1}{8}\tau Rn5 = \tfrac{1}{56}\tau^2 Rn6 \quad (21)$$

Substituting Eqs. (14 and 16–18) into Eq. (5) and approximating $1/(\rho_T + Rn1)^2$ by $[(1/\rho_T^2) - (2Rn1/\rho_T^3)]$, the resulting equation is obtained by omitting the terms that contain Rn^3, Rn^2, or the product of Rn's. It is permissible because they are of smaller order of magnitude, and θ_T and ρ_T have their order of magnitude near unity. Therefore,

$$\rho_T'' + Rn3 = \frac{A}{1 - B\tau}\left[M\left(1 - \frac{1}{2}\theta_T^2 - \theta_T Rn4\right) + N\left((\theta_T + Rn4 - \frac{1}{6}\theta_T^3 - \frac{1}{2}\theta_T^2 Rn4\right)\right] + (\rho_T\theta_T'^2 + \theta_T'^2 Rn1 + 2\rho_T\theta_T' Rn5) - \frac{1}{\rho_T^2} + \frac{2Rn1}{\rho_T^3} \quad (22)$$

Eq. (6) can be written as

$$\rho^2\frac{d^2\theta}{d\tau^2} + 2\rho\frac{d\rho}{d\tau}\cdot\frac{d\theta}{d\tau} = \frac{A}{1 - B\tau}\rho\left[N\left(1 - \frac{1}{2}\theta^2\right) - M\left(\theta - \frac{1}{6}\theta^3\right)\right] \quad (23)$$

Applying the same technique in obtaining Eq. (22) to Eq. (23) yields

$$\rho_T^2\theta_T'' + 2\rho_T\theta_T'' Rn1 + \rho_T^2 Rn6 + 2(\rho_T\rho_T'\theta_T' + \theta_T'\rho_T' Rn1 + \rho_T\theta_T' Rn2 + \rho_T\rho_T' Rn5) = \frac{A}{1 - B\tau}\rho_T\left[N\left(1 - \frac{1}{2}\theta_T^2 - \theta_T Rn4\right) - M\left(\theta_T + Rn4 - \frac{1}{6}\theta_T^3 - \frac{1}{2}\theta_T^2 Rn4\right)\right] + \frac{A}{1 - B\tau}Rn1\left[N\left(1 - \frac{1}{2}\theta_T^2\right) - M\left(\theta_T - \frac{1}{6}\theta_T^3\right)\right] \quad (24)$$

The solutions of Eqs. (22) and (24) with the aid of Eqs. (20) and (21) give the values of all Rn's.

It is recalled that, in the previous analysis, approximations were made for $\sin\theta$ and $\cos\theta$ by the first two terms of their infinite series, respectively. In order to complete the estimate of accuracy, it is necessary to investigate the errors introduced due to these approximations.

Considering Eq. (17), because $Rn4 \ll 1$ and hence $\sin Rn4 \simeq Rn4$, $\cos Rn4 \simeq 1$, and

$$\left.\begin{aligned}\sin\theta &= \sin(\theta_T + Rn4) = \sin\theta_T + Rn4\cos\theta_T\\ \cos\theta &= \cos(\theta_T + Rn4) = \cos\theta_T - Rn4\sin\theta_T\end{aligned}\right\} \quad (25)$$

With Eqs. (25), the following equations, which are equivalent to Eqs. (22) and (24), are obtained from Eqs. (3) and (4):

$$\rho_T'' + Rn3 = \frac{A}{1 - B\tau}[M(\cos\theta_T - Rn4\sin\theta_T) + N(\sin\theta_T + Rn4\cos\theta_T)] + (\rho_T\theta_T'^2 + \theta_T'^2 Rn1 + 2\rho_T\theta_T' Rn5) - \frac{1}{\rho_T^2} + \frac{2Rn1}{\rho_T^3} \quad (26)$$

$$\rho_T^2\theta_T'' + 2\rho_T\theta_T'' Rn1 + \rho_T^2 Rn6 + 2(\rho_T\rho_T'\theta_T' + \theta_T'\rho_T' Rn1 + \rho_T\theta_T' Rn2 + \rho_T\rho_T' Rn5) = \frac{A}{1 - B\tau}\rho_T[N(\cos\theta_T - Rn4\sin\theta_T) - M(\sin\theta_T + Rn4\cos\theta_T)] + \frac{A}{1 - B\tau}Rn1(N\cos\theta_T - M\sin\theta_T) \quad (27)$$

Solving Eqs. (20, 21, 26, and 27) simultaneously yields another set of Rn's, which contain both the truncation error and the error due to the approximations made for $\sin\theta$ and $\cos\theta$. It is noted that the differences between the Rn's obtained from Eqs. (26) and (27) and those from Eqs. (22) and (24) are the errors due the approximations made for $\sin\theta$ and $\cos\theta$.

The accuracy of any truncated powered series decreases as the magnitude of its independent variable increases; thus, for any pre-assigned accuracy, the maximum value of the independent variable could be determined. For the present case the independent variable is τ. After computing the ρ, θ, $d\rho/d\tau$, and $d\theta/d\tau$ at the maximum τ, a new series can be developed by using the previous "maximum τ" as the origin. This procedure can be repeated again and again until the required total τ is reached.

Example

In order to illustrate the application of the present analysis, the following numerical example will be examined. A rocket is originally in a 300-naut mile circular orbit, orbiting counterclockwise around the earth, and is subject to a constant thrusting force of M_0g_0 in the direction of the radius vector r_0. What is the resulting powered flight trajectory if the thrust-on time is 91.32 sec ($\bar{c} = 10^4$ fps)?

From the given data, the initial conditions are $a_0 = 1$, $a_1 = 0$, $b_0 = 0$, $b_1 = 1$, $A = 1$, $B = 2.4880\ 2590$, $N = 0$, and $M = -1$. With these initial conditions, all a's and b's are obtained from Eqs. (12) and (13). Hence

$$\rho_T = 1 - 0.5000\ 00000\tau^2 - 0.4146\ 70983\tau^3 - 0.3491\ 89407\tau^4 - 0.6042\ 09569\tau^5 - 0.9621\ 11645\tau^6 - 1.6781\ 16569\tau^7 \quad (28)$$

$$\theta_T = \tau + 0.5000\ 00000\tau^3 + 0.4146\ 70983\tau^4 + 0.6908\ 56073\tau^5 + 1.1156\ 37114\tau^6 + 1.8725\ 24904\tau^7 \quad (29)$$

and ρ_T', ρ_T'', etc., can be obtained accordingly. Note that all a's and b's should be calculated to a sufficient number of figures after the decimal point because they will affect the estimate of accuracy. For a certain value of τ, ρ_T, ρ_T', etc. can be calculated by Eqs. (28) and (29) and their respective derivatives, and the remainders Rn's then are determined by Eqs. (22) and (24) or Eqs. (26) and (27). Tables 1 and 2 are obtained for $\tau = 0.1$ ($t = 91.32$ sec.)

Now, if the accuracies shown in the last column of Table 2 are satisfied, then Eqs. (28) and (29) with their respective derivatives should be the approximate solution of the powered flight trajectory with known accuracy. For a longer thrust-on time, the computation of a powered flight trajectory can be carried out in steps with a pre assigned accuracy at the end of each step as stated previously The usual short thrust-on time of liquid chemical propellant or solid propellant rocket engine requires only a few steps for the computation of an entire powered flight trajectory; if such is the case, the analytical expressions of various elements of the "last stretch" of the powered flight trajectory still may be valuable to a system engineer because these expressions can be incorporated with the system equations to yield the desired informations.

As was mentioned in the introduction, the flat earth approximation often has been employed in optimizing the range of a ballistic missile, and it is interesting to explore the errors introduced at the burnout for the foregoing example if the flat earth approximation is used in computing the powered flight trajectory during the powered flight. In addition, the case of impulsive thrust also is investigated for the purpose of comparison. The total range angle (angle between the radius vector r_0 and the radius vector passing through the impact point on the earth surface) for each case also is computed.

For the convenience of presentation, the usual X-Y rectilinear coordinates system is adapted by taking the point $r = r_0$, $\theta = 0$ as the origin and the r_0 radius vector as the Y axis. The gravitational acceleration for the flat earth approximation is assumed to be g_0, and the propellant consumed for the impulsive thrust case is the same as that for the other two cases. Table 3 shows the burnout velocities, the burnout location, and the total range angle for the three cases considered.

It is seen from Table 3 that the burnout conditions play a very important role in finding the total range angle. A comparison between case I and case II at the burnout readily can bring out the effect of the assumption of the flat earth: for case I, as the powered flight proceeds, the X component of the gravitational attraction gradually increases, and the resulting V_x should be smaller than that of case II, which assumes no gravitational attraction in the X direction. In the Y direction, the opposite is true, because for case II the gravitational attraction is constant during the powered flight, whereas in case I the gravitational force increases as the rocket proceeds; however, the difference is small since the change in r is small. After examining the burnout velocities, the comparison on the burnout locations between case I and case II can be reasoned easily.

As to case III, the results presented in Table 3 further substantiate the conclusion drawn by the author in his previous paper[5]—any error analysis based on impulsive thrust case is of doubtful value, for the effect of finite thrusting time does yield different conic paths at different thrust levels.

Conclusion

Briefly, the series solutions developed in this paper are valid only over a finite time interval. A second series must be expanded using the terminal points of the first as the initial conditions for the second. In other words, an extended formula for numerical integration has been developed. Thus, the integration should be carried out efficiently because few steps are needed, and the accuracy of each step can be controlled within any preassigned value. No doubt digital computers will be employed in carrying out the series solution; however, in checking out the machine program, one will realize that all algebraic equations are solved readily with given numbers.

As an additional application, the present analysis also can apply to the case of orientated constant thrust acceleration just by taking $B = 0$ and A as a pure number, where Ag_0 is the constant thrust acceleration due to the thrust during the powered flight.

References

[1] Tsien, H. S., "Take-off from satellite orbits," Jet Propulsion **23**, 233–236 (1953).

[2] Benny, D. J., "Escape from a circular orbit using tangential thrust," ARS J. **28**, 167–169 (1958).

[3] Lawden, D. F., "Orbital escape from a circular orbit," Astronaut. Acta **IV**, 218–233 (1958).

[4] Wang, K., "Estimate of effect of large thrust on Hohmann-type transfers," ARS J. **32**, 642–643 (1962).

[5] Zee, C. H., "The effect of finite thrusting time in orbital maneuvers," AIAA J. **1**, 60–64 (1963).

[6] Fried, B. D. and Richardson, J. M., "Optimum rocket trajectories," J. Appl. Phys. **27**, 955–961 (1956).